Lecture 6. **Typical determinate perturbations (actions) and corresponding responses**

6.1 Typical determinate perturbations (actions) and corresponding responses

1. *The Unit Step Function*

The mathematical description of *the Unit Step Function* is of the form:

$$\theta_{in}(t) = \begin{cases} 1(t), & at \;\; t \geq 0; \\ 0, & otherwise \end{cases} \qquad (1)$$

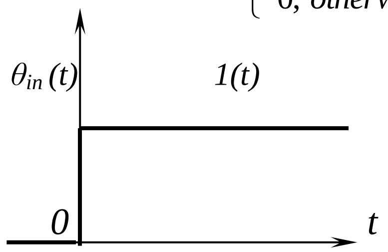

Fig. 2.6. The unit step function

In practice, the unit step function represents instantaneous signal value change, for example, momentary electric generator load change.

The unit step function response, often called *the Step Response*, is *a transient function h(t)*. Physically, *h(t)* is a transient signal observable at the output of a link, when the unit step function occurs at the input of the link under zero initial conditions.

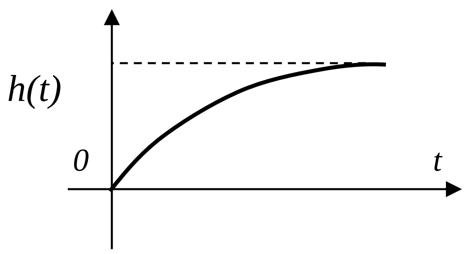

Fig. 2.7. Transient at the output of the first order aperiodic link

The Laplace transform of the unit step function: $L\{1(t)\} = \frac{1}{s}$.

The Laplace transform of the step response: $L\{h(t)\} = H(s)$.

The transfer function of a link is by definition $W(s) \stackrel{\Delta}{=} \frac{\theta_{out}(s)}{\theta_{in}(s)}$.

Solving for θ_{out} gives us $\theta_{out}(s) = W(s)\theta_{in}(s)$. In case $\theta_{in}(s) = 1(t)$ we have $H(s) = W(s)\frac{1}{s}$.

Mathematically, *transient function* is defined as solution of differential equation under zero initial conditions.

2. *The Impulse function*

The unit impulse function, or the unit pulse, denoted as *δ(t)* (delta-function) is a derivative of the unit step function, $\delta(t) \overset{\Delta}{=} 1'(t)$. The delta-function is equal to zero everywhere except the point *t=0,* where it tends to infinity. It is a generalized form of a high-amplitude, short-duration pulse.

The mathematical description it is:

$$\theta_{in}(t) = \delta(t) = \begin{cases} 0, & at \ \ t \neq 0 \\ \infty, & otherwise \end{cases} \qquad (2)$$

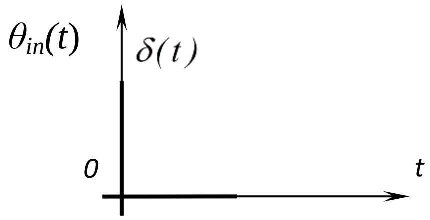

Fig. 2.8. The delta-function *δ(t)*

One of the basic properties of the delta-function is that

$$\int_0^{\infty} \delta(t)dt = 1,$$

e.g., its graph covers very narrow region of a unit area about *t = 0.*

Examples: air-pocket, short-duration short circuit current, momentary high stress on a motor axle.

The Laplace transform of the delta-function: $L\{\delta(t)\} = 1$.

Physically, the delta-function response at zero initial conditions is *the weight function.* Mathematically, *the weight function* is a derivative of the step response transient function:

$$K(t) \overset{\Delta}{=} \frac{dh(t)}{dt} \ . \qquad (3)$$

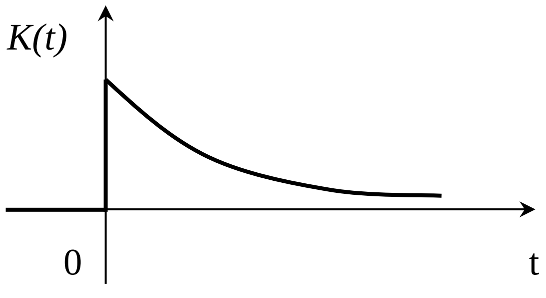

Fig. 2.9. The weight function

The delta-function transform is constant, so, from the transfer function $K(s)=W(s)\cdot 1$. This is an extremely important property used in control theory. Solving the identification problem experimentally, $\delta(t)$ is applied to the input of a system and $K(t)$ is obtained at the output; then after substituting $W(s)=K(s)$ and applying the inverse Laplace transformation a differential equation of the system is obtained, e.g. $\theta_{out}(s)=W(s)\theta_{in}(s)$; $L^{-1}[\theta_{вout}(s)]=L^{-1}[W(s)\theta_{in}(s)]$.

The step response and the weight characteristics are called time characteristics.

3. *The Harmonic (Sinusoidal) Perturbation*

Mathematically, it has the form:

$$\theta_{in}(t)=a\sin\omega t \qquad (4)$$

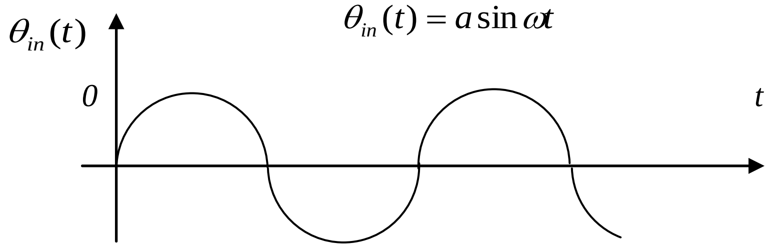

Fig. 2.10. The harmonic input function

The response of a system is considered *only in steady-state*, after the transient have already took place, and is called *the Generalized Frequency Characteristic (Response)*. It can be obtained after application of the Fourier Transformation to (4). Formally, the generalized frequency characteristic can be obtained from the transfer function $W(s)$ substituting $s=j\omega$:

$$W(s)\big|_{s=j\omega}=W(j\omega)=A(\omega)e^{j\varphi(\omega)}, \qquad (5)$$

where $A(\omega)\overset{\Delta}{=}|W(j\omega)|$ is the magnitude of $W(j\omega)$ and is called *the Gain-Frequency Characteristic (GPFC)***;** $\varphi(\omega)$ is called *the Phase-Frequency Characteristic (PhFC)*.

Note, however, that the same equation (5) can be rearranged in other way:

$$W(j\omega)=A(\omega)e^{j\varphi(\omega)}=\operatorname{Re}W(j\omega)+j\operatorname{Im}W(j\omega),$$

where $A(\omega)=|W(j\omega)|=\sqrt{\operatorname{Re}^2(W(j\omega))+\operatorname{Im}^2(W(j\omega))}$; $\varphi(\omega)\overset{\Delta}{=}arctg\,\dfrac{\operatorname{Im}(W(j\omega))}{\operatorname{Re}(W(j\omega))}$.

Conclusion: above we have considered three basic types of determinate input perturbations and corresponding link responses. Important thing to mention about is that *a particular system property is determined by relations between its own and* *input signals time characteristics*. The next topic for discussion is the frequency characteristics of dynamic systems and their properties.

6.2 Frequency-domain analysis (characteristics)

1) Consider generalized frequency characteristic in complex plane, which is also called *Gain-Phase Frequency Characteristic (GPhFC)***.**

$$GPhFC:\ W(s)\Big|_{s=j\omega} = W(j\omega) = A(\omega)e^{j\varphi(\omega)} = \mathrm{Re}W(j\omega) + j\,\mathrm{Im}W(j\omega).$$

It represents geometrical position of vectors' ends (*a hodograph curve* or *locus*) corresponding to frequency transfer function (5) having frequency changing from zero to positive infinity. Real part $\mathrm{Re}W(j\omega)$ (fig. 2.11) is placed on "*x*" axis, imaginary part $\mathrm{Im}W(j\omega)$ is placed on "*y*" axis, $0 \le \omega \le +\infty$. At negative frequency change $-\infty \le \omega < 0$ flip image is produced, since the characteristic is symmetrical about 0.

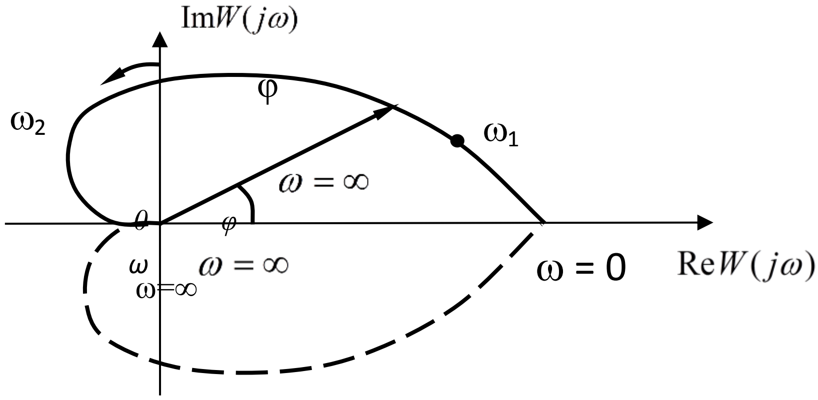

Fig. 2.11. Gain-Phase Frequency Characteristic

2) *Gain-Frequency Characteristic* shows how a link transmits signals of various frequencies. By definition,

$$GFC:\ A(\omega) \overset{\Delta}{=} |W(j\omega)| = \sqrt{\mathrm{Re}^2(W(j\omega)) + \mathrm{Im}^2(W(j\omega))}.$$

$A(\omega)$ is a number describing input-to-output amplitude ratio $\dfrac{A_{out}(\omega_i)}{A_{in}(\omega_i)}$; ω_{cp} is cutoff frequency; $0 \le \omega \le \omega_{cp}$ is frequency bandwidth.

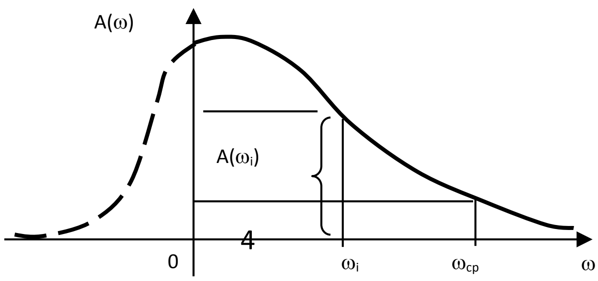

Fig. 2.12. Gain-Frequency Characteristic

GFC (fig. 2.12) is plotted as frequency changes from 0 to $+\infty$, i.e. $0 \le \omega \le +\infty$; negative change gives flip image as the characteristic is symmetrical about 0, i.e. $-\infty \le \omega < 0$.

3) *Phase-Frequency Characteristic*. By definition,

$$PhFC: \ \varphi(\omega) \overset{\Delta}{=} arc\ tg \frac{\operatorname{Im} W(j\omega)}{\operatorname{Re} W(j\omega)}.$$

Phase-Frequency Characteristic represents phase shifts introduced by a link under various frequencies. As a rule, $\varphi(\omega)$ is always lagging (of negative value); naturally phase lag increases with frequency.

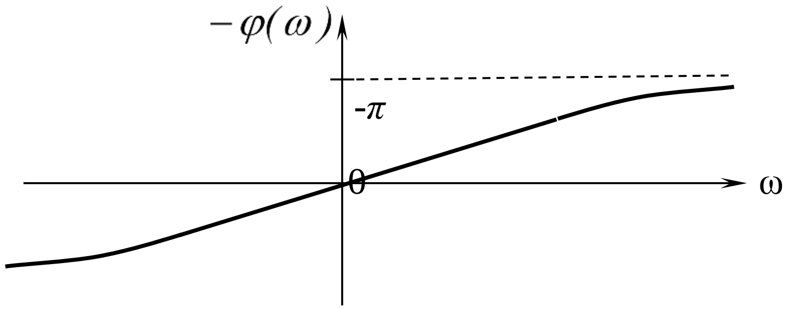

Fig. 2.13. Phase-Frequency Characteristic (*PhFC*)

This characteristic is plotted exactly as GFC and PhFC, with one minor exception: in control theory there is an agreement that for PhFC negative phase shift is placed on positive "*y*" axis, and vice versa.

4) *Logarithmic Frequency Characteristic*

Let us take the logarithm of frequency transfer function (2.35):

$$\ln W(j\omega) = \ln\left\{A(\omega)e^{j\varphi(\omega)}\right\} = \ln A(\omega) + j\varphi(\omega).$$

For practical purposes it is more convenient to use common logarithm and prepare *Logarithmic Gain-Frequency (LgGFC)* (fig. 2.15) and *Logarithmic Phase-Frequency (LgPhFC) Characteristics* (fig. 2.16) separately. For LgGFC preparation we need to find $L(\omega) \overset{\Delta}{=} 20\lg|W(j\omega)| = 20\lg A(\omega)$ (common logarithm). Unit of $L(\omega)$ is a decibel, and unit of frequency logarithm is a decade.

LgGFC is plotted in the following coordinate system (fig. 2.14):

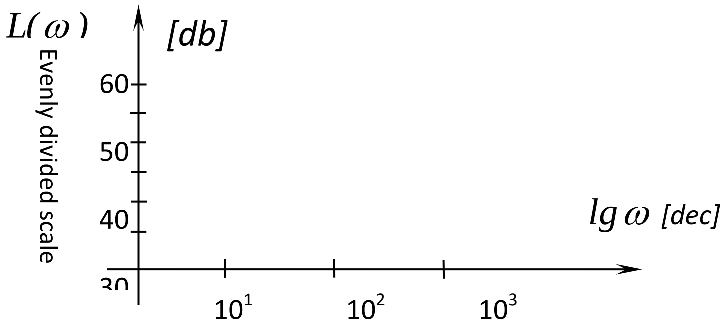

Fig. 2.14. LgGFC coordinate system

Notes:

a) frequency ω is plotted on *"x"* axis in *logarithmic scale,* not in *lg(ω);*

b) since *lg(0)* = -∞ than *"y"* axis can be placed anywhere (is floating); usually it is placed in such a way that the whole graph will lie to the right of it.

The unit of logarithm increment is one decade, i.e. tenfold of frequency. *Asymptotical* LgGFC is plotted as polyline; knee frequency $\omega_i = \frac{1}{T_i}$ corresponding to a salient point is called *connecting (mate) frequency* ($\forall i = \overline{1,n}$).

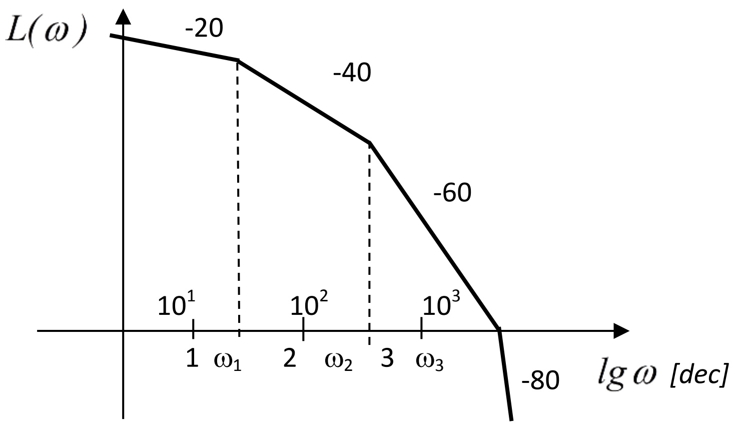

Fig. 2.15. Logarithmic Gain-Frequency Characteristic (*LgGFC*)

Several useful properties of LgGFC deserve should be mentioned:

a) compactness of presentation;

b) simplicity of linear characteristics approximation.

Logarithmic phase-frequency characteristic is a frequency logarithm dependence of φ(ω). This characteristic is not used in practice, since it is dual of PhFC (fig. 2.16).

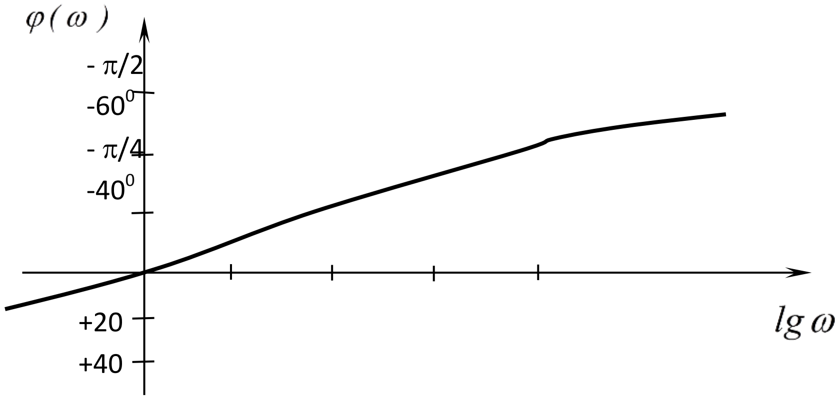

Fig. 2.16. Logarithmic Phase-Frequency Characteristic

The next topic shows characteristics typical dynamic elements in time and frequency fields. Further we will show these characteristics only for aperiodic link of the second order. You must obtain the same characteristics for other typical dynamic elements yourself.

Aperiodic Link of the second order

Mathematical description has the following form:

$$T^2\ddot{\theta}_{out} + 2T\dot{\theta}_{out} + \theta_{out}(t) = K\theta_{in}(t).$$

The transfer function is

$$W(s) = \frac{K}{T^2 s^2 + 2Ts + 1} = \frac{K}{(1+T_1 s)(1+T_2 s)}.$$

Defining equation is

$$(1+T_1 s)(1+T_2 s) = 0; \quad s_1 = -\frac{1}{T_1}, \quad s_2 = -\frac{1}{T_2}.$$

Now *time characteristics* follow.

Transient function (fig. 2.17):

$$h(t) = K[1-(c_1 e^{s_1 t} + c_2 e^{s_2 t})] = 1-(c_1 e^{-\frac{1}{T_1}t} + c_2 e^{-\frac{1}{T_2}t}), \text{ at } K\equiv 1.$$

t	$h(t)$
0	$K(1-(c_1+c_2)$; $K \neq 0$; $c_1+c_2 = 1$;
T_i	< 1;
∞	1.

Weight function (fig. 2.18):

$$K(t) \overset{\Delta}{=} \frac{dh}{dt} = \frac{c_1}{T_1} e^{-\frac{1}{T_1}t} + \frac{c_2}{T_2} e^{-\frac{1}{T_2}t}.$$

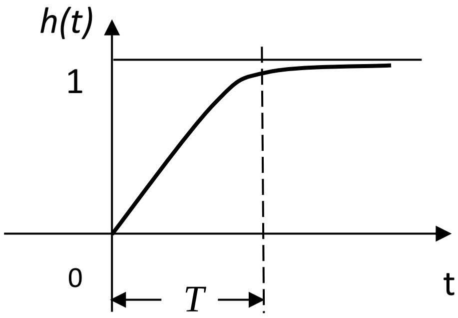

Fig. 2.17. Transient function link

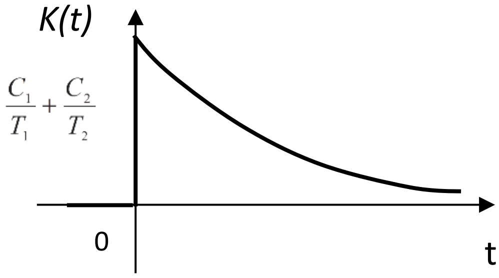

Fig. 2.18. Weight function link

Here go frequency characteristics.

Gain-phase frequency characteristic *GPhFC* (fig. 2.19):

$$W(j\omega) = W(s)\big|_{s=j\omega} = \frac{K}{(1 + jT_1\omega)(1 + jT_2\omega)} = \frac{K(1 - jT_1\omega)(1 - jT_2\omega)}{(1 + T_1^2\omega^2)(1 + T_2^2\omega^2)} =$$

$$= \frac{K(1 - T_1T_2\omega^2)}{(1 + T_1^2\omega^2)(1 + T_2^2\omega^2)} - j\frac{K(T_1 + T_2)\omega}{(1 + T_1^2\omega^2)(1 + T_2^2\omega^2)}.$$

ω	*Im*	*Re*
0	0	*K*
$\frac{1}{T_i}$	$-\frac{K}{2}$	0
∞	0	0

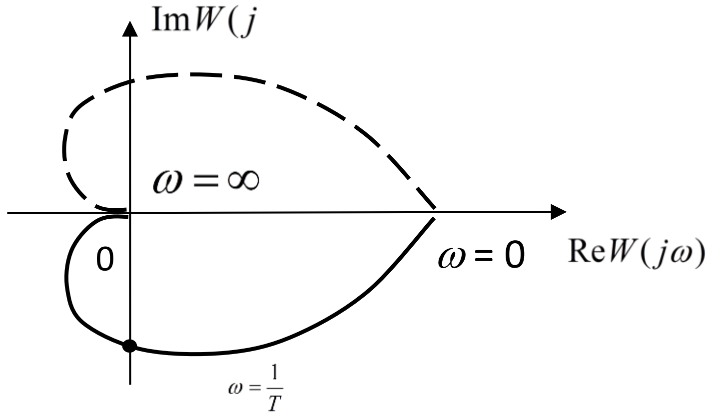

Fig. 2.19 GPhFC link

Gain-frequency characteristic *GFC* (fig. 2.20):

$$A(\omega) \overset{\Delta}{=} \sqrt{\mathrm{Re}^2 W(j\omega) + \mathrm{Im}^2 W(j\omega)} = \frac{K\sqrt{(1 - T_1T_2\omega^2)^2 + \omega^2(T_1 + T_2)^2}}{(1 + T_1^2\omega^2)(1 + T_2^2\omega^2)}.$$

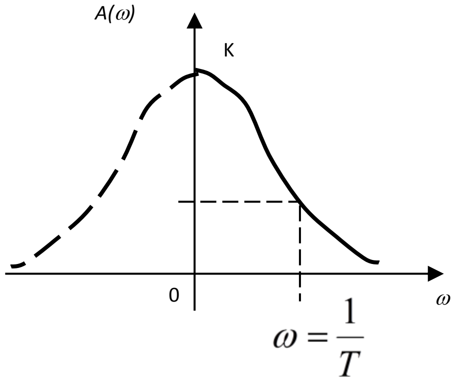

Fig. 2.20. GFC link

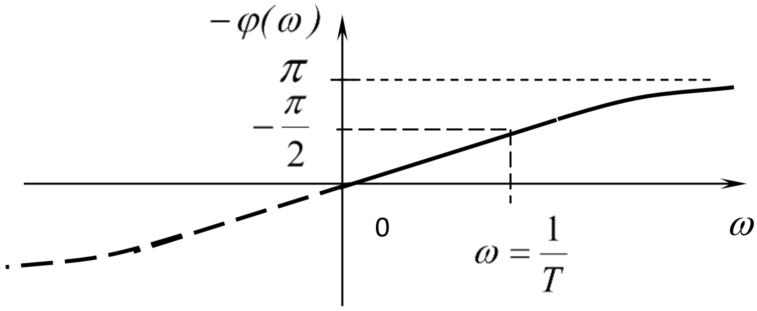

Fig. 2.21. PhFC link

Phase-frequency characteristic *PhFC* (fig. 2.21):

$$\varphi(\omega)\overset{\Delta}{=} arctg\frac{\operatorname{Im}W(j\omega)}{\operatorname{Re}W(j\omega)} = arctg\left(-\frac{\omega(T_1+T_2)}{1-T_1T_2\omega^2}\right).$$

Logarithmic gain-frequency characteristic *LgGFC* (fig. 2.22):

$$L(\omega)\overset{\Delta}{=}20\lg|W(j\omega)| = 20\lg A(\omega) = 20\lg\left|\frac{K}{(1+T_1s)(1+T_2s)}\right|_{s=j\omega}.$$

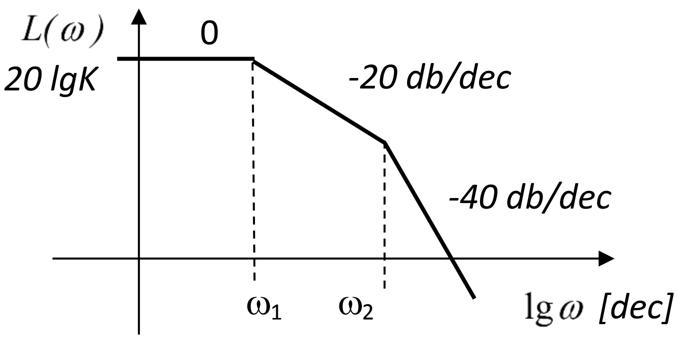

Fig. 2.22. LgGFC link

Connecting frequencies are: $\omega_1 = \frac{1}{T_1}$, $\omega_2 = \frac{1}{T_2}$; $\omega_1 < \omega_2$ since $T_1 > T_2$.

A rule for *asymptotic LgGFC plotting*: it is drawn as a polyline with slope ±*20n* [db/dec] decibel per decade, where "*n*" is power of "*s*" in open-loop system transfer function *W(s)* .

In conclusion it is necessary to mark that by means of the typical dynamic links (connected sequentially, parallel, etc.) it is possible to provide any dynamic system. Using rules of algebra of structural conversions, we receive transfer function of the whole system.

Properties of system are not the amount of properties of the separate links which are logging in though, naturally, each link surely contributes to the general properties of the system. In it the integrity of system is shown.